Assignment 13

We follow the notations in Chapter 4 of our notes. This assignment is concerned with tangent and normal vectors of curves and surfaces.

1. Let $\gamma : (a, b) \to \mathbb{R}^2$ be a parametric curve and $t \in (a, b)$. The tangent space of Γ at $p = \gamma(t)$ is defined to be the one dimensional space spanned by the vector $\gamma'(t)$.

   (a) Show that the definition of the tangent space is independent of reparametrization (so it is a geometric property).

   (b) A normal vector is a vector (a, b) that is perpendicular to the tangent vector. Let $\gamma(s)$ be parametrised by arc-length, that is, $|\gamma'(s)| = 1$. Show that $n(s) = (-\gamma_2'(s), \gamma_1'(s))$ is a unit normal vector.

2. Optional.

   (a) Let $\gamma : [0, L] \to \mathbb{R}^2$ be a closed parametric curve. Show that there exists an open set G containing Γ such that for each $(x, y) \in G$, there exist some s and a such that $(x, y) = \gamma(s) + an(s)$.

   (b) Suppose further that γ is simple, that is, $\gamma(s_1) \neq \gamma(s_2)$ all $s_1 < s_2 \in [0, L)$ and C^2. Show there exists a small $a_0 > 0$ such that G can be taken to be $G = \{(x, y) : \text{dist}((x, y), \Gamma) < a_0\}$ and for each $(x, y) \in G$, the s and a in (a) is unique.

3. (a) Let $\sigma : R \to \mathbb{R}^3$ be a parametric surface and $p = \sigma(s, t) \in \Sigma$. The tangent space of Σ at p is the two dimensional vector subspace spanned by $\partial\sigma/\partial s, \partial\sigma/\partial t$. Show that the tangent space at p is independent of reparametrisation. Propose a definition of reparametrization of surfaces.

   (b) Let $\gamma : I \to \Sigma$ be a parametric curve on Σ passing $p = \gamma(t)$. Show that $\gamma'(t)$ belongs to the tangent space of Σ at p.

   (c) In fact, show that every tangent vector at Σ arises in the way as is described in (b).

   (d) Let Σ be the locus of a C^1-function $f(x, y, z) = 0$ with nonvanishing gradient. Show that the gradient vector $\nabla f(x, y, z)$ is perpendicular to all tangent vectors at (x, y, z). In other words, it points to the normal direction.

4. Here is a typical case of Lagrange multipliers. Consider the constrained minimization problem
$$\inf\{f(x, y, z) : (x, y, z) \text{ satisfies } g(x, y, z) = 0\} ,$$
where f and g and C^1-functions. Let p_0 be a local minimum of this problem and suppose that $\nabla g(p_0) \neq (0, 0, 0)$. Show that there exists some scalar λ such that $\nabla f(p_0) = \lambda \nabla g(p_0)$. Suggestion: Show that $\nabla f(p_0)$ lies in the normal direction of the surface of $g = 0$.

5. Consider the Cauchy problem
$$\frac{dx}{dt} = f(x), \quad x(t_0) = x_0 \in \mathbb{R}^3 .$$
(We have rewritten (2.3) in Notes by replacing x, y by t, x respectively and assume f is independent of t.) Suppose that $f(x) \cdot x = 0$ and x_0 lies on the unit sphere. Show that $x(t)$ remains on the sphere and the maximal solution of this Cauchy problem exists for all $t \in (-\infty, \infty)$.